2-dimensional normal PDF

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k-dimensional normal joint PDF:

$$\frac{1}{\sqrt{(2\pi)^k|\Sigma|}}\exp\left(-\frac{1}{2}(x-\mu)^T\Sigma^{-1}(x-\mu)\right)$$

$k=2$, $\Sigma = \begin{vmatrix} \sigma_x^2 & \sigma_{12} \\ \sigma_{12} & \sigma_Y^2 \end{vmatrix}$ $\qquad \rho = \dfrac{\sigma_{12}}{\sigma_x\sigma_Y}$ $\qquad \rho^2\sigma_x^2\sigma_Y^2 = \sigma_{12}^2$

这叫作方差.协方差矩阵

用ρ表示

$$|\Sigma| = \sigma_x^2\sigma_Y^2 - \sigma_{12}^2 = \sigma_x^2\sigma_Y^2 - \rho^2\sigma_x^2\sigma_Y^2 = \sigma_x^2\sigma_Y^2(1-\rho^2)$$

$$= \sigma_x^2\sigma_Y^2(1-\rho^2)$$

$\begin{vmatrix} a & b \\ c & d \end{vmatrix}^{-1} = \dfrac{1}{ad-bc}\begin{vmatrix} d & -b \\ -c & a \end{vmatrix}$

同样地,用ρ表示.

$$\Sigma^{-1} = \frac{1}{\sigma_x^2\sigma_Y^2-\sigma_{12}^2}\begin{vmatrix} \sigma_Y^2 & -\sigma_{12} \\ \sigma_{12} & \sigma_x^2 \end{vmatrix} = \frac{1}{\sigma_x^2\sigma_Y^2(1-\rho^2)}\begin{vmatrix} \sigma_Y & -\rho\sigma_x\sigma_Y \\ -\rho\sigma_x\sigma_Y & \sigma_x \end{vmatrix}$$

$$\left[(2\pi)^2\sqrt{\sigma_x^2\sigma_Y^2(1-\rho^2)}\right]^{-1}\cdot\exp\left\{-\frac{1}{2}\underbrace{[X-\mu_x, Y-\mu_Y]}_{①}\cdot\frac{1}{\sigma_x^2\sigma_Y^2(1-\rho^2)}\underbrace{\begin{vmatrix} \sigma_Y & -\rho\sigma_x\sigma_Y \\ -\rho\sigma_x\sigma_Y & \sigma_x \end{vmatrix}}_{②}\right.$$

$$\Rightarrow \left[(2\pi)^2\sqrt{\sigma_x^2\sigma_Y^2(1-\rho^2)}\right]^{-1} \qquad \begin{bmatrix} X-\mu_x \\ Y-\mu_Y \end{bmatrix} ③$$

$$\frac{1}{\sigma_x^2\sigma_Y^2(1-\rho^2)}\left[(X-\mu_x)^2\sigma_Y - (Y-\mu_Y)\rho\sigma_x\sigma_Y(X-\mu_x),\ -\rho\sigma_x\sigma_Y(X-\mu_x)(Y-\mu_Y) + \sigma_x(Y-\mu_Y)^2\right]$$

$\sigma_x^2\sigma_Y^2$ 乘进去

$$\frac{1}{(1-\rho^2)}\left(\frac{(X-\mu_x)^2}{\sigma_x^2} - \frac{2\rho\cdot(X-\mu_x)(Y-\mu_Y)}{\sigma_x\sigma_Y} + \frac{(Y-\mu_Y)^2}{\sigma_Y^2}\right)$$

PDF:

$$\Rightarrow \left((2\pi)\cdot\sigma_x\sigma_Y\sqrt{1-\rho^2}\right)^{-1}\exp\left\{-\frac{1}{2}\cdot\frac{1}{(1-\rho^2)}\left[\frac{(X-\mu_x)^2}{\sigma_x^2} - \frac{2\rho\cdot(X-\mu_x)(Y-\mu_Y)}{\sigma_x\sigma_Y} + \frac{(Y-\mu_Y)^2}{\sigma_Y^2}\right]\right\}$$

证明 correlation 为 ρ.

$$\rho_{XY} = \frac{cov(X,Y)}{\sigma_x\sigma_Y} = \frac{E(X-\mu_x)(Y-\mu_Y)}{\sigma_x\sigma_Y}$$

$$= \int(\sigma_x\sigma_y)^{-1}\cdot(x-\mu_x)(y-\mu_y)\cdot f(x,y)\,dx\,dy.$$

$$= \int\left(\frac{x-\mu_x}{\sigma_x}\right)\left(\frac{y-\mu_Y}{\sigma_y}\right)f(x,y)\,dx\,dy = \rho$$

want to show

$s = \dfrac{x-\mu_x}{\sigma_x}\cdot\dfrac{y-\mu_y}{\sigma_y}$; $t = \dfrac{x-\mu_x}{\sigma_x}$

$x = \sigma_x t + \mu_x$ $\qquad x = \sigma_x t + \mu_x$ $\qquad \dfrac{dx}{dt} = \sigma_x$ $\quad \dfrac{dx}{ds} = 0$

$\dfrac{y-\mu_y}{\sigma_y} = \dfrac{s}{t}$ $\Rightarrow$ $y = \dfrac{s}{t}\sigma_y + \mu_y$ $\qquad \dfrac{dy}{ds} = \dfrac{\sigma_y}{t}$

change of variable

$\therefore |J| = \left|\dfrac{\sigma_x\sigma_y}{t}\right|$

$$= \int s\cdot\left((2\pi)\cdot\sigma_x\sigma_Y\sqrt{1-\rho^2}\right)^{-1}\exp\left\{-\frac{1}{2}\frac{1}{1-\rho^2}\left(t^2 - 2\rho\cdot s + \left(\frac{s}{t}\right)^2\right)\right\}|J|\,ds\,dt$$

$$\frac{t^4 - 2\rho s t^2 + s^2}{t^2}$$

$$\left(t-\frac{s}{t}\right)^2 + 2s - 2\rho s$$

$$= \int s\left(2\pi\cdot\sigma_x\sigma_Y\sqrt{1-\rho^2}\right)^{-1}\exp\left\{-\frac{1}{2}\frac{1}{1-\rho^2}\left[\left(t-\frac{s}{t}\right)^2 + 2s(1-\rho)\right]\right\}\cdot\frac{\sigma_x\sigma_Y}{t}\,ds\,dt$$

$$\int s\frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{1-\rho^2}}\frac{1}{\sqrt{2\pi t^2}}\exp\left\{-\frac{1}{2}\cdot\frac{\left(t-\frac{s}{t}\right)^2}{1-\rho^2}\right\}\exp\left\{-\frac{s(1-\rho)}{1-\rho^2}\right\}.\quad ds\,dt$$

$$= \int\frac{1}{\sqrt{2\pi t^2}}\cdot\exp\left\{\quad\right\}\cdot s\cdot\frac{1}{\sqrt{2\pi}\sqrt{1-\rho^2}}\cdot\exp\left\{-\frac{1}{2}\left(\frac{\left(t-\frac{s}{t}\right)^2}{1-\rho^2}\right)\right\}ds\,dt$$

Σ normal.